

The quantum-like description of the dynamics of party governance in the US political system

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Abstract

This paper is devoted to the application of the mathematical formalism of quantum mechanics to social (political) science. By using the quantum dynamical equations we model the process of decision making in US elections. The crucial point we attempt to make is that the voter's mental state can be represented as a superposition of two possible choices for either republicans or democrats. However, reality dictates a more complicated situation: typically a voter participates in two elections, i.e. the congress and the presidential elections. In both elections he/she has to decide between two choices. This very feature of the US election system requires that the mental state is represented by a 2-qubit state corresponding to the superposition of 4 different choices (e.g. for republicans in the congress; for the president as a democrat). The main issue of this paper is to describe the dynamics of the voters' mental states taking in account the mental and socio-political environment. What is truly novel in this paper is that instead of using Schrödinger's equation to describe the dynamics in an absence of interactions, we here apply the quantum master equation. This equation describes quantum decoherence, i.e., resolution from superposition to a definite choice.

1 Introduction

For the last ten years, the mathematical formalism of quantum theory has been actively applied outside the domain of quantum physics. We have seen numerous applications in decision making (both in cognitive and social science), economics and also finance. See for instance Acacio de Barros and Suppes (2009) [1], Asano et al. (2010) [3], Bruza et al. (2005 [6], 2009a [7], 2009b [8]); Busemeyer et al. (2006a [10], 2006b [11]); Cheon et al. (2006 [13], 2010 [14]); Choustova (2007 [15]), Pothos et al. (2009) [35], Franco (2009) [23], Haven (2006 [24], 2008a [25], 2008b [26], 2009 [27]) and La Mura (2008) [32].

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Recently the *quantum-like* (QL) approach started to be explored in political science. Some of the QL features of the behavior of voters in the US political system were discussed in Zorn and Smith (2011) [45]. The authors start with a comparison of the notions of state separability in conventional models of party governance and in quantum information theory (see Zorn and Smith (2011) [45]) and they then show that the QL model might provide a more adequate description of the voters' state space – 'mental space'. The authors present a strong motivation of the usage of the complex Hilbert space as the voters' 'mental space.'

In this paper we present a QL-model describing the dynamics of the voters' state (as represented in the complex Hilbert space). First, we consider what we could call 'a free QL-dynamics', when a voter² is not under the pressure of mass media and the social environment. By applying the quantum approach we describe the dynamics of her state by using an analogue of the Schrödinger equation. A simple mathematical analysis implies that Alice's preferences encoded in her state-vector ('mental wave function') fluctuate *without* stabilization to the definite state. Hence, such a dynamics can describe the unstable part of the electorate: those voters who have no firm preferences. In quantum physics, stabilization and damping of fluctuations is a typical consequence of interaction with the environment³. We apply this approach to the problem of the stabilization of fluctuations of voters' preferences.

An essential part of the paper is devoted to the analysis of the applicability of quantum dynamics to a social system (e.g. a voter) which is coupled to the social environment. The main problem is that the exact quantum dynamics of a system coupled to the physical environment is extremely complicated. Therefore, to simplify matters, typically a quantum Markov approximation is applied. This approximation is applicable under a number of non-trivial conditions (see Ingarden et al. (1997) [28]). Our aim is to translate these conditions into the language of social science and to analyze their applicability to the dynamics of voters' preferences. In this connection the quantum Markovian dynamics, especially via the quantum master equation, can model (approximately) voters' preference dynamics. Our approach is based on the quantum master equation which describes the interaction of a social system with a 'social bath'. We use a very general framework which can be applied to a variety of problems in politics, social science, economics, and finance. The main problem of any concrete application is to analyze the conditions of applicability of the quantum master equation (the quantum Markov approximation) to the corresponding problem in decision making.

² Following the tradition of quantum information theory, we call such a voter 'Alice'. In game theory 'Alice' is also often used.

³ Such environment in physics is also known as 'bath'.

We remark that the work of Fiorina (1996) [21] played an important role in the motivation of the quantum model based on the use of entangled quantum states (see Zorn and Smith (2011) [45] for the two institutional choices in U.S. politics – the congress and the presidency). Zorn and Smith (2011) [45] also present a detailed analysis of the inter-relation between classical and quantum models. Such an analysis is very important to attract the interest of mainstream researchers in decision making to quantum models. For such researchers, the applications of the quantum formalism to social science may on prima facie be considered as quite exotic. Therefore, in this paper, we begin with an extended section in which we compare classical and quantum probabilistic approaches to decision making. Our aim is not only to stress the differences, but also to find the commonalities. Our findings argue for an important degree of similarity between quantum and subadditive probabilistic descriptions of decision making. We also emphasize the vital role of contextuality.

2 Inter-relation between quantum and traditional models of decision making

2.1 (Non-) Bayesian approach

One of the basic tools of probabilistic investigations in psychology, cognitive science, economics and finance is Bayesian analysis (see De Finetti (1972) [17] and Kreps (1988) [31]) which allows for a process of mental updating of probabilities (objective or subjective depending on the interpretation⁴) on the basis of newly collected statistical data. Bayesian probability can be distinguished to be objective (independent of the individual who makes a decision) or subjective, that is to say, related to the personal belief of an individual (see De Finetti (1974) [17]). The objective probabilities represent the choice that rational agents should make in the light of an objective situation and updating occurs as a consequence of the appearance of any new event (see Chalmers (1999) [12]). By this approach the agents are supposed to distribute the prior probabilities equally on the basis of some principle of indifference. In particular, the Bayesian approach plays an important role in classical decision making (see De Finetti (1972) [17]). We stress that this method is a part of conventional (‘classical’) probability theory based on Kolmogorov’s axiomatics (1950) [30]. The Bayes formula for conditional probabilities is:

$$P(A|B) = P(AB)/P(B); \tag{1}$$

⁴ There are two ‘camps’ around the interpretations of Bayesian probabilities divided in so called ‘objectivists’ and ‘subjectivists’. See Chalmers (1999) [12].

where $P(B) \neq 0$. The law of total probability forms an integral part of the classical Bayesian approach. Let us consider the law of total probability in the simplest situation. Consider an event B and its complement \bar{B} and assume that the probabilities of both these events are positive. Then, for any event A , the following formula (of total probability) holds:

$$P(A) = P(B)P(A|B) + P(\bar{B})P(A|\bar{B}). \quad (2)$$

We note that for *quantum probabilities*, the law of total probability is violated! In general, the difference between the left-hand and right-hand sides of (2) is nonzero. This difference is nothing else than the influence of the interference term, which plays a fundamental role in quantum theory (as well as in classical physical wave theories). The quantum analog of the law of total probability has the form:

$$P(A) = P(B)P(A|B) + P(\bar{B})P(A|\bar{B}) + 2\cos \theta \sqrt{P(B)P(A|B)P(\bar{B})P(A|\bar{B})}. \quad (3)$$

Depending on the sign of $\cos \theta$ one observes constructive ($\cos \theta > 0$) or destructive ($\cos \theta < 0$) interference. In the first case the probability to observe some phenomenon increases so much that it cannot be explained by the laws of classical probability theory. In the second case one similarly finds a ‘mystical’ decreasing of probability (e.g., probabilities $P_1 = P_2 = 1/2$ can result in a zero probability, $P_{12} = 0$). In the case $\cos \theta = 0$ the quantum formula of total probability (the formula containing thus an interference of probabilities) is reduced to the classical law of total probability. This is a very important point of transition from usage of the classical probabilistic model to the quantum probabilistic model.

By decreasing the absolute value of interference coefficient, the latter can be transformed into the former (as the coefficient vanishes). Thus the quantum probabilistic models in cognitive science, psychology, and social science are natural extensions of the classical models. If the deviation of the left-hand side of equation (3) from the right-hand side is relatively small, we can ignore the interference contribution and proceed with the classical law of total probability.

In summary, we can think of the quantum approach of decision making as a natural generalization of the Bayesian approach which is based on the transition from the classical formula of total probability to its quantum analogue.

2.2 *Subadditive probability in social science, psychology, behavioral economics and finance*

The following questions naturally arise when considering the use of equation (3) as a new tool in decision making:

- (1) Is the departure from equation (2) to equation (3) a totally new step in the development of probabilistic modeling in social science?
- (2) Are there other conventional social models based on departures from the laws of classical probability?

Surprisingly for those who argue for the exceptional novelty of the quantum approach to social problems, the answer is ‘yes’. In mainstream studies in cognitive science, psychology, behavioral economics and finance, non-classical probability has been actively used during many years.

Comparing the quantum approach with the traditional non-classical probabilistic approaches is not a straightforward task. In QL models the *violation of the law of total probability* is considered as the crucial point. However, the majority of traditional non-classical models are not based on the aforementioned violation of equation(2), but rather on an application of *subadditive probabilities*. Hence, the *violation of the law of additive probability* has already been actively discussed in social science. Khrennikov and Haven (2007) [29] indicate (p. 23) that “when experiment participants have to express their degree of beliefs on a $[0, 1]$ interval, probabilistic additivity will be violated in many cases and subadditivity obtains. See Bearden et al. (2005) [4] for a good overview.” Khrennikov and Haven (2007) [29] continue as follows (p. 23-24): “Bearden et al. (2005) [4] also indicate that such subadditivity has been obtained with experiment participants belonging to various industry groups, such as option traders for instance (Fox et al. (1996) [22]). The key work pertaining to the issue of subadditivity in psychology is by Tversky and Koehler (1994) [41] and Rottenstreich and Tversky (1997) [36]. Their theory, also known under the name of ‘Support Theory’ is in the words of Tversky and Koehler (1994) [41] ‘...a theory in which the judged probability of an event depends on the explicitness of its description.’ In other words, it is not the event which is important as such but its description. In Tversky and Koehler (1994) [41] the authors highlight the ‘current state of affairs’...on the various interpretations that subjective probability may have. Amongst the interpretations is Zadeh’s (1978) [44] possibility theory and the upper and lower probability approach of Suppes (1974) [38]. The paper of Dubois and Prade (1998) [18], also mentioned in Tversky and Koehler (1994) [41], provides for an excellent overview on non-additive probability approaches.”

To couple QL models based on the violation of the law of total probability and Bayesian probability, with traditional studies based on subadditive probabilities, we need to recall that the mathematical derivation of the formula of total probability is based on the additivity of probability and the Bayes formula for conditional probabilities. Therefore there are two possible sources of violation of equation (2): i) subadditivity and ii) the non-Bayesian definition of conditional probability. Both these sources exhibit themselves in QL-models. Hence, the subadditivity of probability is an important common point of the QL and traditional (based on non-classical probability) approaches. Moreover, many experts in quantum physics especially stress the role of subadditivity of quantum probability as the main source of quantum interference (Feynman and Hibbs (1965) [20]).

The QL approach can be considered as a special mathematical model describing the usage of subadditive probability in social science. It is not clear whether any social science based model with subadditive probability can be embedded in the QL-approach. The quantum probabilities have a very special structure: they are based on complex probability amplitudes, vectors from a complex linear space and probability is obtained from a squared complex amplitude. *It is not clear whether any subadditive probability from the aforementioned social science based models can be represented in this way.* Nevertheless, even if it might imply the loss of generality, the use of the linear space representation simplifies the operation with probabilities. Furthermore, it provides us with a possibility to use a powerful mathematical apparatus of quantum mechanics in an interdisciplinary way.

In this paper, we intend to explore quantum dynamical equations. We stress that the form of these equations depend very much on whether interaction with the environment is taken into account or neglected. Here we are merely interested in the application of quantum dynamics to the modeling of the evolution of the mental state of a human being interacting with an extremely complex social environment. The complexity of the actual environment is so high that it strongly influences the decision making process of an individual, finally implying a resolution from superposition of his/her mental states. In quantum terms a decoherence takes place.

2.3 Savage sure thing principle and disjunction effect

Bayesian probability according to Maher (2010) [33] (p. 120) “explicates a kind of rationality we would like our choices to have...” Correspondingly, the ‘absolute rational choice’ Maher (2010) (p.120) refers to, can be understood as “the maximization of expected utility.” The Bayesian updating of probabilities and the validity of the law of total probability have a direct coupling with the

problem of rationality in decision making. von Neumann and Morgenstern's (1944) [42] expected utility theory, and Savage's sure thing principle (Savage (1954) [37]) postulate a complete rationality (i.e. a maximization of one's own payoff and the minimizing of one's own losses). Savage (1954) [37] (p. 21) proposed the so called 'Sure thing principle (STP)', denoting that: "If a person would not prefer [a decision] f to g , either knowing that the event B obtained, or knowing that the event \bar{B} obtained, then he does not prefer f to g [whether knowing or not if the event B or \bar{B} happened]." Savage (1954) [37] illustrates the validity of the principle with an example of a businessman, who considers whether to buy some property before the presidential elections or not. Savage (1954) (p. 21) describes the situation of a businessman who is uncertain if the Republicans or Democrats will win the election campaign. He decides that he would buy the property if the Republicans win, but also he decides that he should buy the property even if the Democrats win. By taking the decision to buy in any case (for example the decision g), it is natural to assume that the businessman will buy the property being uncertain of whether the Republicans (event B) or Democrats will win (event \bar{B}). The principle could be statistically represented with help of the formula of total probability (equation (2)), where the events B and \bar{B} are assigned some probability and the decision A (here depicted as g , to be consistent with Savage's symbols) would be a conditional probability of B and \bar{B} , so that the exact statistical probability for the possible decision g could be obtained. In this illustration we see that the conditional probability of g would be equal to one (i.e. there is 100% confidence about the purchase of a house). According to Croson (1999) [16] the event B can be as well i) an exogenous risk: the uncertainty about the state of nature (e.g. the property purchase) as well as ii) a strategic risk: an uncertainty about the choice of a strategic opponent (e.g. a competitor starts a price war). Croson (1999) [16] describes such a pattern of decision making as 'consequential reasoning', as the individual considers the consequences (for instance the amount of the payoffs) before considering a particular action.

Savage's Sure thing Principle has been regarded as a foundation axiom for decision making in economics. Kreps (1998) [31] (p. 120) called Savage's principle the "crowning glory of choice theories". However, many experiments, such as Allais (1953) [2], Tversky and Shafir (1992) [40], Croson (1999) [16] proved that economic decision makers in general tend to violate the Savage sure thing principle and expected utility theory. For example, in a Prisoner Dilemma type game experiment, violation of the rationality postulate of Savage's sure thing principle was found in experiments performed by Tversky and Shafir (1992) [40] and later repeated by Croson (1999) [16] and Busemeyer et al. (2006) [10]. Traditionally, this game is played in three conditions. In the 'unknown' condition the player acts without knowing the opponent's action. In the known 'defect condition', the player knows that the opponent has defected before he/she acted. In the known 'cooperate condition' the player knows that the opponent has cooperated, before he/she acted. See also Tversky and Shafir

(1992) [40] and Pothos and Busemeyer (2009) [35].

We cite Tversky and Shafir (1992) [40] (p. 309) “The subjects... played a series of prisoners dilemma games, without feedback, each against a different unknown opponent supposedly selected at random from among the participants. In this setup the rate of cooperation was 3% when subjects knew that the subject knew that the opponent has defected and 16% when they knew that the opponent has cooperated. However, when the subjects did not know whether their opponent had cooperated or defected (as is normally the case of the game) [condition of uncertainty]) the rate of cooperation rose to 37%.” This experiment showed that when the players are unaware of their opponents actions, they do not behave rationally as they are supposed to do in a conventional prisoners dilemma game. This anomaly in behavior occurred in other games of the Prisoners Dilemma type and also in Hawaiian vacation experiments. The basic effect those experiments have in common is referred to by Tversky and Shafir (1992) [40] and Croson (1999) [16] as the ‘disjunction effect’. Busemeyer et al. (2006) [?] show that the disjunction effect is equivalent to the violation of the law of total probability. Since this law is violated by QL models, all such models in social science exhibit the disjunction affect.

2.4 Contextuality

In the quantum community there is still no consensus on the basic roots of ‘quantum mysteries’; in particular, the grounds for the violation of the laws of classical probability theory. One hundred years after the creation of quantum mechanics (the 1920’s-1930’s starting with the founders of quantum mechanics : Bohr, Heisenberg and Einstein), the intensity of debates about its foundation have not abated. We may even claim the debates are more intense. One of the possible sources of the quantum mysteries is the notion of contextuality. The viewpoint that the results of quantum observations depend crucially on the measurement context was proposed by Niels Bohr, who emphasized that we are not able to approach the micro world (with the aid of our measurement devices) without bringing essential disturbances into its state. The quantum systems are too sensitive to the measurement apparata. The context of measurement plays an essential (depending on the interpretation, even crucial) role in forming the result of our measurement. According to the fundamental interpretation of quantum mechanics, the ‘Copenhagen interpretation’, quantum systems do not have objective properties which exist independently of ‘questions’ asked to these systems in the context of measurement. Says Suppes (1974) [38] (p. 171-172): “Any time we measure a microscopic object by using macroscopic apparatus we disturb the state of the microscopic object and, according to the fundamental ideas of quantum mechanics, we cannot hope to improve the situation by using new methods of measurement that

will lead to exact results of the classical sort for simultaneously measured conjugate variables.”

The contextual viewpoint is attributed to the origin of non-classical probabilistic behavior of quantum systems and is very attractive for those who already apply or aim to apply a quantum formalism in other domains outside physics. It is important to stress that the contextual interpretation of quantum mechanics is more ‘innocent’ than other essentially more exotic viewpoints, such as the quantum non-locality concept or the ‘many worlds’ interpretations. The majority of people working in cognitive science and psychology would not accept a possibility of non-local interactions between human beings, e.g. through a splitting of reality in many worlds.

The concept of contextuality is a well known feature of cognitive systems. We also see the origin of non-Bayesian (‘irrational’⁵) decision making in the contextuality of observations performed for mental quantities, including self-observations. Hence, the value of the subjective probability does not exist independently of the measurement context, only whilst ‘asking about someone’s preferences’ including ourselves, we create them.

For example, in semantics studies context is treated by representing it as cue words, or co-appearing words. This semantic contextuality (well known and actively explored in traditional semantic models) was used as the starting point for the development of QL models of word recognition (see Bruza et al. (2005 [6], 2009a [7], 2009b [8])). We also remark that contextual models of reasoning play an important role in artificial intelligence (see f.i. Giunchiglia (1993) [19], McCarthy (1993) [34]).

We now come back to the problem of rationality in decision making. We remark that contextuality of reasoning is closely coupled with the so called ‘framing effect’. Kreps (1988) [31] remarks (p. 197) that “the way in which a decision problem is framed or posed can affect the choices made by decision makers.” According to Tversky and Kahnemann (1981) [39] the term ‘decision frame’ refers (p. 453) “to the decision-maker’s conception of the acts, outcomes and contingencies associated with a particular choice.” One of the most important contributions of the QL approach to the problem of contextual reasoning is the recognition of the existence of incompatible contexts and the use of well developed quantum tools for testing incompatibility, such as Heisenberg’s uncertainty relation or the violation of Bell’s inequality (see f.i. Khrennikov and Haven (2007) [29]). In particular, in the Prisoner’s Dilemma game the contexts C_{known} (the decision of the partner is known), and C_{unknown} (information is absent), are incompatible. Consequently, the QL approach is about:

- (1) the violation of the Sure Thing Principle,

⁵ For a comparison, please see section 2.3. (especially Maher (2010) [33])

- (2) ‘irrational’ decision making,
- (3) non-Bayesian decision making, and
- (4) the usage of subadditive probability.

All these problems have already been widely discussed in traditional approaches to cognitive science, psychology, behavioral economics and finance. The QL approach is just one of the mathematical models which accurately describes all of the above effects.

Finally, we point to one of the pioneer papers that assigned quantum-like contextuality to the measurement of belief in decision making theories. Suppes (1974) [38] conjectured that general concepts taken from quantum mechanics could provide for the measurement of belief. He also explained the importance of the particular measurement context, by asserting that (p. 172): ‘it is a mistake to think of beliefs as being stored in some fixed and inert form in the memory of a person. When a question is asked about personal beliefs, one constructs a belief coded in a belief statement as a response to the question. As the kind of question varies, the construction varies, and the results vary.’

We could articulate that the notion of measurement context, borrowed from quantum mechanics can be regarded as one of the promising theories of measurement of belief.

3 Description of election campaign by the theory of open quantum systems

With the help of the above mentioned features of QL models we now attempt to describe the dynamics of the process of decision making within the problem setting of party governance in the US-type two party system. This system allows voters to cast partisan ballots in two contests: executive and legislative. By so doing they can thus choose for instance ‘Republican’ in one institutional choice setting and ‘Democratic’ in the other (see Zorn and Smith (2011) [45]).

It is well known from physics that the quantum state dynamics are described by Schrödinger’s equation. This type of dynamics is unitary. Roughly speaking it is combined of a family of rotations and in principle, this family can be infinite. Pothos and Busemeyer (2009) [35] applied this equation to model the dynamics of the process of decision making in games of the Prisoner’s Dilemma type. However, it is questionable whether we can describe the dynamics of voters’ expectation by the Schrödinger’s equation. This equation describes the dynamics of an isolated system, i.e., a system which does not interact with the environment. A voter in the context of the election campaign definitely cannot be considered as an isolated social system. She, say Alice, is in permanent

contact with mass media (whether TV or internet). Such an influence of the environment induces random fluctuations of opinions and choices in Alice’s mind.

For the purposes of our research, we are interested in the ‘unstable’ part of the electorate which is composed of citizens who have no concrete opinions and who will make their electoral choice very close to the actual day of the elections (see Zaller and Feldman (1992) [43]).

If Alice could be considered as an isolated social system, then the only possibility to describe a transition from the mental state of superposition of choices to the state corresponding to the concrete choice was to use the projection postulate of quantum mechanics (the so called ‘von Neumann postulate’). This state reduction process, from superposition to one of its components, is called *the state collapse*⁶. Such collapse is imagined as an instantaneous (the jump-type) transition from one state to another. The state collapse might be used to describe the situation in which Alice makes her choice precisely at the moment of completing the voting bulletin. This type of behavior cannot be completely excluded from consideration, but such a case is probably not statistically significant. Moreover, mainstream quantum mechanical thought will tell us that the state collapse occurs when an isolated system driven by Schrödinger’s equation interacts practically instantaneously with a measurement device. Thus when Alice is totally isolated from the election campaign, she is suddenly asked to make her choice. It is evident that the process of decision making for the majority of the ‘unstable population’ in the electorate differs in essential ways from this collapse-type behavior.

Therefore, let us take more seriously the role which the social environment plays in the process of decision making. We apply to social science the theory of *open quantum systems*, i.e., systems which interact with a large thermostat (‘bath’). Since a bath is a huge physical system with millions of variables (the complexity of the “social bath” around an American citizen who will cast his/her vote in the election campaign is huge), it is in general impossible to provide a reasonable mathematical description of the dynamics of a quantum system interacting with such a bath. Physicists proceed under a few assumptions which allow then for the possibility to describe those dynamics in an approximate way. In quantum physics the interaction of a quantum system with a bath is described by a quantum version of the master equation for Markovian dynamics. The quantum Markovian dynamics are given by the *Gorini-Kossakowski-Sudarshan-Lindblad* (GKSL) equation. See e.g. Ingarden et al. (1997) [28] for details. This GKSL equation is the most popular approximation of quantum dynamics in the presence of interaction with a bath.

⁶ The state collapse is considered as one of the main mysteries of quantum physics. This notion is still a subject of intensive debate.

We briefly remind the origins of the GKSL-dynamics. The starting point is that the state of a composite system, a quantum system s combined with a bath, is a pure quantum state, complex vector Ψ . The evolution of Ψ is described by Schrödinger's equation. This is an evolution in a Hilbert space of a huge dimension, since a bath has so many degrees of freedom. The existence of the Schrödinger dynamics in the huge Hilbert space has a merely theoretical value. Observers are interested in the dynamics of the state ϕ_s of the quantum system s . The next fundamental assumption in the derivation of the GKSL-equation is the Markovian character of the evolution, i.e. the absence of long term memory effects. It is assumed that interaction with the bath destroys such effects. Thus, the GKSL-evolution is a Markovian evolution. Finally, we point to the condition of the 'factorizability' of the initial state of a composite system (a quantum system coupled with a bath), $\Psi = \phi_s \otimes \phi_{\text{bath}}$, where \otimes is the sign of the tensor product. Physically factorization is equivalent to the absence of correlations⁷. One of the distinguishing features of the evolution under the mentioned assumptions is the existence of one or few *equilibrium points*. The state of the quantum system s stabilizes to one of such points in the process of evolution: a pure initial state, a complex vector ψ_s , is transformed into a mixed state, a density matrix $\rho_s(t)$ (classical state without superposition effects).

In contrast to the GKSL-evolution, the Schrödinger evolution does *not* induce stabilization. Any solution different from an eigenvector of the Hamiltonian will oscillate for ever. Another property of the Schrödinger dynamics is that it *always* transfers a pure state into a pure state, i.e., a vector into a vector: quantumness if it was originally present in a state (in the form of superposition) cannot disappear in the process of a continuous dynamical evolution. The transition from quantum indeterminism to classical determinism can happen only as the result of the collapse of the quantum state.

On the one hand, in our model of the decision making for party governance we would like to avoid the usage of the state collapse. On the other hand, to make a decision, Alice has to make a transition from a quantum to a classical representation of her preferences. We note that in quantum physics all experimentally obtained information is classical as well. The GKSL-evolution provides for such a possibility (and without 'quantum jumps'). Alice's mental state evolves in a smooth way (fluctuations exist but they are damped) to the final classical decision state.

⁷ At the beginning of evolution; later they are induced by the interaction term of Hamiltonian – the generator of evolution.

4 Matching of assumptions of applicability of the quantum master equation with conditions of the modern election campaign

We now list the social conditions corresponding to the above mentioned physical conditions. This will allow us for a possibility to apply the GKSL-equation:

- **(COMPL)** *complexity*: the social environment (election bath) influencing a voter has huge complexity
- **(FREE)** *freedom*: the mental state of a society under consideration is a pure QL state, i.e., a superposition of various opinions and expectations;
- **(DEM)** *democracy*: the feedback reaction of a voter to the election bath is negligibly small, it cannot essentially change the mental state of the bath;
- **(SEP)** *separability*: before the start of the election campaign a voter was independent of the election bath;
- **(MARK)** *Markovness*: a voter does not use a long range memory on interaction with the election bath to update her state.

We surely need to make some comments on those assumptions.

- (1) The assumption **(COMPL)**, *complexity*, is definitely justified. Nowadays an election campaign has huge information complexity: the richness of media sources accounts for such complexity. We can even speculate that the proposed QL model is more adequate than say 50 years ago: the phenomenal increase of information complexity makes the usage of the (quantum, quantum-like) open systems approach more reasonable.
- (2) The **(FREE)**, *freedom*, can be interpreted as guaranteeing the freedom of political opinions. The opposite to the **(FREE)**-society, is a totalitarian society where its mental state is a classical state in which all superpositions have been resolved (collapsed).
- (3) The **(DEM)**, *democracy*, encodes the democratic system: one voter cannot change the mental state of society in a crucial way.
- (4) The **(SEP)**, *separability*, describes a sample of voters who are not that interested in politics: they will determine their positions through an interaction with the election bath during the election campaign. This part of the electorate is the most interesting from the point of view of political technologies.
- (5) The **(MARK)**-assumption, *Markovness*, also reflects the fact that voters under study are not that interested in politics. They do not spend a lot of time analyzing the dynamics of the election campaign. However, they are not isolated from the election bath; they watch TV, read newspapers and use the internet. From a pragmatic point of view, they unconsciously update their mental states each day by taking into account recent news.

Remark 1 (Markovness) *We remark that the Markovness of the dynamics*

may induce the impression that voter's preferences would fluctuate forever. However, this is not the case. The mathematical formalism of quantum mechanics implies that quantum Markovian fluctuations stabilize to steady solutions. In physics, this theoretical prediction was confirmed by numerous experiments. Although the social counterparts of physical assumptions seem to be natural and this motivates the applicability of our theoretical model, the final justification can come only from the testing of our hypothesis by experimental data. This is a very complex problem.

Remark 2 (Decoherence) *In quantum physics the process of transformation of a pure (superposition-type) state into a classical state (given by a diagonal density matrix) is called decoherence. A proper interpretation of this process is still one of the hardest problems in the foundations of quantum mechanics. Some authors present the viewpoint that superposition is in some way conserved: the disappearance of superposition in a subsystem increases it in the total system. In our model this would mean that the determination of states of voters in the process of interaction with the election bath will transfer political uncertainty into an increase of political uncertainty in society in general, after elections. At the moment it is not clear whether this interpretation is meaningful in social sciences.*

5 Schrödinger's dynamics

The state space of a voter (Alice) can be represented as the tensor product of two Hilbert spaces (each of them is two dimensional). One Hilbert space describes the election to the congress, and we denote it by the symbol $\mathcal{H}_{\text{congress}}$, and another describes the presidential election, denote it by the symbol $\mathcal{H}_{\text{president}}$. In each of them we can select the basis corresponding to the definite strategies $e_1 = |d\rangle, e_2 = |r\rangle$. If Alice was thinking only about the election to congress, her mental state would be represented as the superposition of these two basis vectors:

$$\psi_{\text{congress}} = \alpha_c |d\rangle + \beta_c |r\rangle; \quad (4)$$

where α_c, β_c are complex numbers and they are normalized by the condition: $|\alpha_c|^2 + |\beta_c|^2 = 1$. By knowing the representation of equation (4) one can find the probabilities of intentions to vote for democrats and republicans in the election to the congress:

$$p_{\text{congress}}(d) = |\alpha_c|^2, \quad p_{\text{congress}}(r) = |\beta_c|^2. \quad (5)$$

However, the quantum dynamics of the state, $\psi_{\text{congress}}(t)$, in the absence of interactions with the political bath (environment), see equation (12) below – ‘social Schrödinger equation’, is such that the probabilities $p_{\text{congress}}(d; t)$,

$p_{\text{congress}}(r; t)$ fluctuate. Therefore, even if Alice wanted to vote for republicans at $t = t_0$, in the process of mental evolution she will change her mind many times.

In the same way, if Alice was thinking only about the election of the president, her mental state would be represented as a superposition of the two basis vectors

$$\psi_{\text{president}} = \alpha_p |d\rangle + \beta_p |r\rangle; \quad (6)$$

where $|\alpha_p|^2 + |\beta_p|^2 = 1$. The corresponding probabilities are given by

$$p_{\text{president}}(d) = |\alpha_p|^2, \quad p_{\text{president}}(r) = |\beta_p|^2. \quad (7)$$

For a moment, let us forget about the quantum model and turn to classical probability theory. Suppose that the classical probabilities $p_{\text{congress}}(d)$, $p_{\text{congress}}(r)$, $p_{\text{president}}(d)$, $p_{\text{president}}(r)$ are given. Furthermore, suppose that voters do not have any kind of correlations between two elections: their choice in the election to the congress does not depend on their choice of the president and vice versa⁸. In this case independence implies factorization of the joint probability distribution:

$$p_{\text{congress,president}}(dd) = p_{\text{congress}}(d)p_{\text{president}}(d), \dots \quad (8)$$

However, in the case of non-trivial correlations between the congress- and president-elections, the factorization condition is violated. In the quantum formalism, the models described by the two Hilbert spaces are unified in the model described by the tensor product of these two spaces. In our case we use the space $\mathcal{H} = \mathcal{H}_{\text{congress}} \otimes H_{\text{president}}$. Its elements are of the form:

$$\psi = \psi_{\text{congress}} \otimes \psi_{\text{president}} \quad (9)$$

which describe the states corresponding to uncorrelated choices in two elections. In quantum information such states are called *separable*. In general a state $\psi \in \mathcal{H}$ cannot be factorized. Nonseparable states describe correlations between choices in the two elections:

$$\psi = c_{dd}|dd\rangle + c_{dp}|dp\rangle + c_{pd}|pd\rangle + c_{pp}|pp\rangle; \quad (10)$$

where $|c_{dd}|^2 + |c_{dp}|^2 + |c_{pd}|^2 + |c_{pp}|^2 = 1$ and $|dd\rangle = |d\rangle \otimes |d\rangle, \dots$. The main point of usage of the quantum formalism is that quantum correlations are not reduced to classical correlations (as described in the framework of the Kolmogorov model). Roughly speaking the quantum correlations can be stronger than the classical correlations. This is the essence of Bell's theorem, Bell (1987) [5]. We also state again that the question of inter-relation between quantum and classical separability in the election framework was studied in Zorn and Smith

⁸ To escape the problem of time fluctuations of probabilities, we may assume that both elections are done at the same time.

(2011) [45]. However, the authors did not appeal directly to Bell's theorem, but to the more delicate condition of quantum (non-) separability. Its role in social science was emphasized by Bruza et al. (2010) [9].

The quantum dynamical equation has the form:

$$i\hbar \frac{\partial \psi}{\partial t}(t) = H\psi(t); \quad (11)$$

where H is the operator of energy, the Hamiltonian, and \hbar is the Planck constant. The mental interpretation of an analog of the Planck constant \hbar is a complicated problem. We shall interpret it as the time scale parameter⁹. Since the usage of the symbol \hbar may be a source of misunderstanding (especially for physical science educated readers), we shall use a new scaling parameter, say τ having the dimension of time (please see the preceding footnote). It determines the time scale of updating of the mental state of Alice during the election campaign. We rewrite the dynamical equation as:

$$i\tau \frac{\partial \psi}{\partial t}(t) = H\psi(t). \quad (12)$$

And we call the operator H , the *decision Hamiltonian*.

The most general Hamiltonian H in the space of mental states in the two-party systems (wherein voters can cast partisan ballots in two contests, executive and legislative) has the form

$$H = H_{\text{stab}} + H_{\text{flip}}, \quad (13)$$

where H_{stab} is the part of the Hamiltonian responsible for the stability of the distribution of opinions about various possible selections of decisions. It is given by

$$H_{\text{stab}} = \lambda_{dd,dd}|dd\rangle\langle dd| + \lambda_{rr,rr}|rr\rangle\langle rr| + \lambda_{dr,dr}|dr\rangle\langle dr| + \lambda_{rd,rd}|rd\rangle\langle rd|. \quad (14)$$

And H_{flip} is the part of Hamiltonian responsible for flipping from one selection of the pair of strategies (for executive and legislative branches) to another. It is given by

$$H_{\text{flip}} = \lambda_{dd,rr}|dd\rangle\langle rr| + \lambda_{rr,dd}|rr\rangle\langle dd| + \lambda_{rd,dr}|rd\rangle\langle dr| + \lambda_{dr,rd}|dr\rangle\langle rd|. \quad (15)$$

To induce a unitary evolution, the Hamiltonian has to be Hermitian. This

⁹ In quantum physics the Planck constant has the dimension of action: time \times energy. In this paper we do not want to speculate on such a controversial topic as “mental energy” (but see however, Choustova (2007) [15]). Therefore, we proceed formally by considering the evolution generator H as a dimensionless quantity.

induces the following restrictions to its coefficients: $\lambda_{dd,dd}, \dots, \lambda_{rd,rd}$ are real and $\bar{\lambda}_{dd,rr} = \lambda_{rr,dd}, \dots, \bar{\lambda}_{rd,dr} = \lambda_{dr,rd}$.

In the absence of the H_{flip} -component, the probabilistic structure of superposition is preserved. Only phases of choices evolve in the rotation-like way, e.g., $|dd\rangle$ evolves as

$$\psi_{dd}(t) = e^{-\frac{i\lambda}{\tau} \lambda_{dd,dd}} |dd\rangle$$

which corresponds to a “rotation” of the strategy dd for the “angle” $\Delta\theta = t\lambda_{dd,dd}$. A larger λ induces quicker rotation. The meaning of such rotations of mental states has to be clarified in the process of the model’s development. We can speculate that the coefficient $\lambda_{dd,dd}$ correspond to the speed of self-analysis (by Alice) of the choice dd .

In the presence of the flipping component H_{flip} the distribution of probabilities of choices of various strategies changes in the process of evolution. Such flipping from one strategy to another makes the state dynamics really quantum. In fact, for political technologies per sé, the most important component is the flipping part of the Hamiltonian. Of course, at the moment we proceed at a very abstract theoretical level. However, one may hope to develop the present QL model to the level of real applications.

Example 3 Suppose that Alice has neither a firm association with democrats nor with republicans, i.e., the diagonal elements of the decision Hamiltonian are equal to zero. Suppose also that the flipping part of the Hamiltonian contains only the transition:

$$|dr\rangle \rightarrow |rd\rangle, \quad (16)$$

which expresses the combination (democrats, republicans) into the combination (republicans, democrats), and vice versa. Let $\lambda_{dr,rd} = \lambda_{rd,dr} = \lambda > 0$. The Schrödinger equation has the form of a system of linear ordinary differential equations. The dynamics of coincidence of choices is trivial: $i\tau \frac{dx_{dd}(t)}{dt} = 0, i\tau \frac{dx_{rr}(t)}{dt} = 0$. Hence, $x_{dd}(t) = x_{dd}(0), x_{rr}(t) = x_{rr}(0)$. However, the presence of a nontrivial transition channel, equation (16), induces fluctuations of Alice’s preferences for choices dr and rd . Here we have the system of two equations:

$$i\tau \frac{dx_{dr}(t)}{dt} = \lambda x_{rd}(t), \quad i\tau \frac{dx_{rd}(t)}{dt} = \lambda x_{dr}(t). \quad (17)$$

Its solutions have the form:

$$x_{dr}(t) = x_{dr}(0) \cos \frac{\lambda t}{\tau} - i x_{rd}(0) \sin \frac{\lambda t}{\tau}, \quad (18)$$

$$x_{rd}(t) = -i x_{dr}(0) \sin \frac{\lambda t}{\tau} + x_{rd}(0) \cos \frac{\lambda t}{\tau}. \quad (19)$$

6 Dynamics in the election bath

In physics the dynamics of a system in a bath is described by the quantum analog of the master equation, the GKSL-equation, see section 3. We write this equation by using the time scaling constant τ , instead of the Planck constant:

$$i\tau \frac{\partial \rho}{\partial t}(t) = [H, \rho(t)] + L(\rho(t)); \quad (20)$$

where L is a linear operator acting in the space of operators on the complex Hilbert space. In the dynamics described by equation (20), density operators are transformed into density operators. The general form of L was found by Gorini, Kossakowski, Sudarshan, and Lindblad (see, for example, Ingarden et al. (1997) [28]). For now, we are not interested in (the rather complex) structure of L . For our applications, it is sufficient to know that it can be expressed through matrix multiplication for a family of matrices. The simplest dynamics of interaction of Alice with the two party election campaign is determined by two matrices V_d and V_r corresponding to advertising of democrats and republicans, respectively. Under natural selection of the matrices H, V_d, V_r any solution of this equation stabilizes to a diagonal density matrix

$$\rho_{\text{decision}} = \text{diag}(p_{dd}, p_{dr}, p_{rd}, p_{rr}). \quad (21)$$

This matrix describes the distribution of firmly established decisions for voting strategies xy , where $x, y = d, r$.

The density matrix ρ_{decision} describes a population of voters who finally determine their choices. Denote the number of people in this population by N . There are then (approximately) $n_{dd} = p_{dd}N$ people in the mental state $|dd\rangle$,...and $n_{rr} = p_{rr}N$ people in the mental state $|rr\rangle$. For example, people in the mental state $|dd\rangle$ have firmly selected to vote for democrats both in the executive and legislative branches. Their decision is stable. From a pragmatic point of view there is no possibility to manipulate opinions of people in this population.

6.1 Comparing mental states given by vectors and density matrices

Consider two populations, say \mathcal{E}_1 and \mathcal{E}_2 . Suppose that in our QL model the first one is described by a pure state

$$\psi = c_{dd}|dd\rangle + c_{dr}|dr\rangle + c_{rd}|rd\rangle + c_{rr}|rr\rangle; \quad (22)$$

and the second one by the density matrix given by equation (21). Moreover, suppose that the complex amplitudes given by the coefficients in the expansion

(equation (22)) produce the same probabilities as the density matrix, i.e.:

$$p_{xy} = |c_{xy}|^2. \quad (23)$$

One may ask: “What is the difference?” At first sight there is no difference at all, since we obtain the same probability distribution of preferences. However, the distributions of mental state in ensembles \mathcal{E}_1 and \mathcal{E}_2 are totally different. All people in \mathcal{E}_1 are in the same state of indeterminacy (superposition) ψ . They are in doubt. They are ready to change their opinion (to create a new superposition of opinions). The \mathcal{E}_1 is a proper population for political manipulations. To the opposite of population \mathcal{E}_1 , the population \mathcal{E}_2 consists of people who have already resolved their doubts. Their mental states have already been reduced to states of the form xy , i.e. definite choices¹⁰.

6.2 Independence from the initial state

The general theory of quantum master equations implies that for some important open system dynamics, the limiting probability distribution *does not depend on the initial state!* This mathematical fact has important consequences for our QL model of elections. It tells us that in principle it is possible to create such a quantum open system dynamics (voters interacting with some election bath) such that the desired state ρ_{decision} would be obtained *independently* of the initial mental state of Alice. This theoretical result may play an important role in QL election technologies. Even if a quantum master equation does not have the unique limiting state, there are typically just a few of them. In this case, we can split the set of all pure states (the unit sphere in the complex Hilbert state space) into clusters of voters. For each cluster, we can predict the final distribution of decisions.

We shall illustrate the above discussion by numerical simulation¹¹ of the dynamics of preferences of voters interacting with **the** “election environment”.

Example 4 *We consider only the two dimensional submodel of the general four dimensional model corresponding to a part of the electorate which have “double preferences” – democrats in one of the elections and republicans in another election. So, we reduce the modeling to the subspace with the basis $|dr\rangle, |rd\rangle$. It is assumed that at the beginning (i.e., before interaction with the ‘election environment’) voters are in a superposition of the basic states:*

$$|\psi\rangle = c_1|dr\rangle + c_2|rd\rangle, \quad |c_1|^2 + |c_2|^2 = 1. \quad (24)$$

¹⁰ Please see the discussion under equation (21).

¹¹ Using Mathematica software.

We also assume that in the absence of interaction with the ‘election campaign’ the state of preferences fluctuations are driven by the Schrödinger dynamics considered in Example 1. In the matrix form the corresponding Hamiltonian can be written as

$$\mathcal{H} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}; \quad (25)$$

where $\lambda > 0$ is the parameter describing the intensity of flipping from dr to rd and vice versa. The simplest perturbation of this Schrödinger equation is given by the Lindblad term of the form given by Ingarden et al. (1997) [28]:

$$C\rho C^* - (C^*C\rho + \rho C^*C)/2 = C\rho C^* - \frac{1}{2}\{C^*C, \rho\};$$

where C^* denotes the operator which is the Hermitian adjoint to the operator C . As always in quantum formalism,

$$\{U, V\} = UV + VU,$$

which denotes the anticommutator of two operators U, V . We select the operator C by using its matrix in the basis $|dr\rangle, |rd\rangle$:

$$C = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix};$$

hence,

$$C^* = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix};$$

where the parameter λ is responsible for interaction between the voter’s state. For simplicity, the ‘election campaign’ is selected in the same way as in the Hamiltonian (25). Thus, we proceed with the quantum master equation:

$$\frac{d\rho}{dt}(t) = -i[\mathcal{H}, \rho(t)] + C\rho(t)C^* - \frac{1}{2}\{C^*C, \rho(t)\}. \quad (26)$$

We present the dynamics corresponding to symmetric superposition,

$$c_1 = c_2 = \frac{1}{\sqrt{2}}. \quad (27)$$

See Fig. 1. Strongly asymmetric superposition

$$c_1 = \sqrt{0.9}, c_2 = \sqrt{0.1}. \quad (28)$$

See Fig. 2.

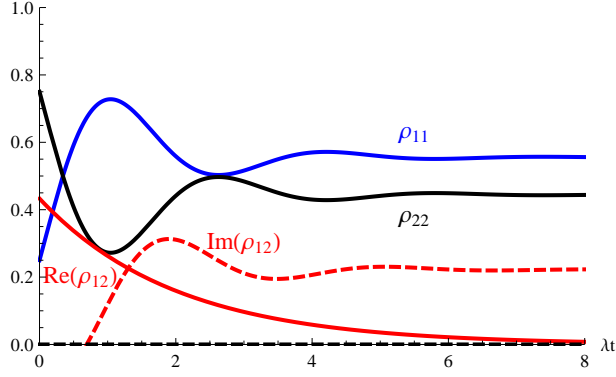


Fig. 1. Stabilization of the matrix elements of the density operator; the initial state is symmetric superposition of state $|dr\rangle$ and $|rd\rangle$.

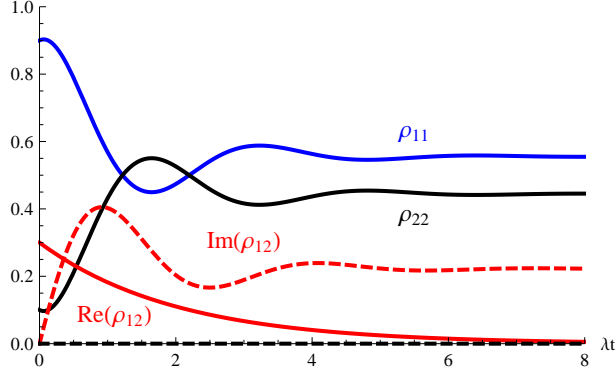


Fig. 2. Stabilization of the matrix elements of the density operator; the initial state is strongly asymmetric superposition of states $|dr\rangle$ and $|rd\rangle$.

The interaction with the ‘election environment’ plays a crucial role. Strong oscillations of the dynamics, given by equations (18), (19) in the absence of interaction with the ‘election bath’ are quickly damped and the matrix elements $\rho_{11} \equiv \rho_{dr,dr}$, $\rho_{22} \equiv \rho_{rd,rd}$, $\rho_{12} \equiv \rho_{dr,rd}$, and $\bar{\rho}_{12} = \rho_{21} \equiv \rho_{rd,dr}$ stabilize to the definite values. Thus the preferences of population of voters who were in fluctuating superposition of choices stabilize under the pressure of the ‘election bath’. We selected such a form of interaction between a voter and the ‘election bath’ such that both initial states, the totally symmetric state, i.e., no preference to dr nor rd , and the state with very strong preference for the dr combination in votes to congress and of president, $p(dr) = 0.9, p(rd) = 0.1$, induce dynamics with stabilization to the same density matrix ρ_{lim} . This example demonstrates the power of the social environment which, in fact, determines the choices of voters.

In the ρ_{lim} the elements $\rho_{dr,dr} \approx 0.6, \rho_{rd,rd} \approx 0.4$ determine corresponding probabilities $p(dr) \approx 0.6, p(rd) = 0.4$. Under the pressure of the social environment those who started with a superposition as indicated in equation (27) increase the dr -preference and those who started with the superposition

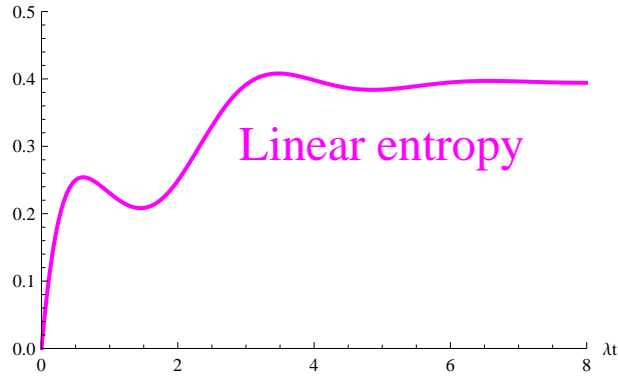


Fig. 3. Stabilization of linear entropy: the initial state is symmetric superposition of states $|dr\rangle$ and $|rd\rangle$.

in equation (28) decrease this preference, and the resulting distribution of choices is the same in both populations (with the initial state (27) and with the initial state (28)).

We stress that manipulation by the preferences described by the dynamics in equation (26) is sufficiently smooth. Those dynamics are an extension of the ‘free thinking’ dynamics given by the Schrödinger equation, the first term in the right-hand side of equation (26). Hence, in this model the social environment does not prohibit internal fluctuations of individuals, but instead damps them to obtain a ‘peaceful’ stabilization.

We emphasize that the degree of quantum uncertainty decreases in the process of evolution. One of the standard measures of uncertainty which is used in quantum information theory is given by so called *linear entropy* (see Ingarden et al. (1997) [28]) defined as:

$$S_L = 1 - \text{Tr}\rho^2.$$

For a pure state (which has the highest degree of uncertainty), the linear entropy $S_{L,\min} = 0$. It increases with degeneration of purity in a quantum state and it approaches its maximal value $S_{L,\max} = 0,5$ for a maximally mixed state. Here we consider the two dimensional case; in the general case $S_{L,\max} = 1 - 1/d$, where d is the dimension of the state space. The dynamics of linear entropy corresponding to the initial states as per equations (27) and (28), respectively, are presented in Fig. 3 and Fig 4.

We see that **the** entropy behaves in different ways, but finally it stabilizes to the same value $S_L \approx 0.4$. **This** value corresponds to a very large decreasing of purity – uncertainty of the superposition type.

Numerical simulation demonstrated that, for other choices of pure initial states, the density matrix and the linear entropy stabilize to the same val-

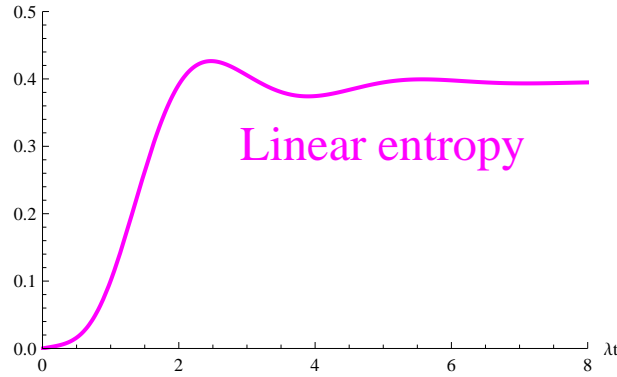


Fig. 4. Stabilization of linear entropy: the initial state is strongly asymmetric superposition of states $|dr\rangle$ and $|rd\rangle$.

ues. Our conjecture is that it may be possible to prove theoretically that this is really the case. However, at the moment we have only results of numerical simulation supporting this conjecture.

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